Input-Output Linearization of General Nonlinear Processes

Michael A. Henson
Dale E. Seborg
Department of Chemical and Nuclear Engineering
University of California
Santa Barbara, CA 93106

Many common processes such as distillation columns, chemical reactions, and pH neutralizations are inherently nonlinear. Until recently, model-based control strategies for nonlinear processes have been based on local linearization and linear controller design based on the linearized model. For highly nonlinear processes, linear feedback controllers must be detuned significantly to ensure their stability, which often severely degrades performance. In recent years, differential geometric methods have been developed that allow exact linearization of the nonlinear model that is independent of the operating point (Isidori, 1989). Linear controllers can then be designed for the equivalent linear system. Existing techniques are of two types: 1. those that linearize the input-state map (Jakubczyk and Respondek, 1980; Hunt et al., 1983) and 2. those that linearize the input-output map.

Several design methods based on input-output linearization have recently been developed for nonlinear process control problems. Methods such as generic model control (Lee and Sullivan, 1988), reference system synthesis (Bartusiak et al., 1989), and internal decoupling (Balchen et al., 1988) are implicitly based on differential geometric concepts (Henson and Seborg, 1990). Although these methods are applicable to models in which the input variables may appear nonlinearly, they are restricted to models in which the rate of change of the output is directly coupled to the manipulated input. Such models are said to have a relative degree of one. Other techniques, such as globally linearizing control (Kravaris and Chung, 1987), are explicitly based on concepts from differential geometry. These techniques are not restricted to models of relative degree one but are only applicable to models which are linear in the input variables.

In this paper, input-output linearization of a more general class of nonlinear processes is investigated. In particular, input variables may appear nonlinearly and the models are not required to have a relative degree of one. Two approaches are considered. The first approach is based on the original process model, while the second is based on an "extended" model that is control linear. Decoupling the output from measured and unmeasured disturbances is also investigated. The two approaches are evaluated via simulation for a chemical reactor model, in which the manipulated coolant flow rate appears nonlinearly.

Input-Output Linearization of General Nonlinear Processes

Existing input-output linearization techniques (Kravaris and Chung, 1987; Daoutidis and Kravaris, 1989) for nonlinear process control are usually based on the models in which the manipulated input and disturbances appear linearly,

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,u + p(x)\,d \\ y &= h(x) \end{aligned} \tag{1}$$

where x is an n-dimensional state vector, d is a p-dimensional disturbance vector, u is a scalar manipulated input, and y is a scalar output. We will consider a more general class of process models, in which the manipulated input and disturbances may appear nonlinearly:

$$\begin{aligned} \dot{x} &= f(x, d, u) \\ y &= h(x) \end{aligned} \tag{2}$$

It is assumed that the process state can be accurately measured or estimated and that the process is minimum phase (Kravaris, 1988; Henson and Seborg, 1990). Several design techniques based on Eq. 2 have been proposed (Lee and Sullivan, 1988; Balchen et al., 1988; Bartusiak et al., 1989), but are restricted to models of relative degree one (Henson and Seborg, 1990).

The Lie derivative of the vector function $f(x)$ and scalar function $h(x)$ is defined as:

$$[L_f^k h](x, d, u) \triangleq \frac{\partial}{\partial x}\{[L_f^{k-1} h]\,(x, d, u)\} f(x, d, u) \tag{3}$$

where $[L_f^0 h](x, d, u) \triangleq h(x)$ is a function of x only. The relative degree r of the manipulated input and relative degree ρ_i of the ith disturbance are defined as:

$$r = \triangleq \min \left\{ k : \frac{\partial}{\partial u} \{[L_f^k h](x, d, u)\} \neq 0 \right\} \quad (4)$$

$$\rho_i \triangleq \min \left\{ k : \frac{\partial}{\partial d_i} \{[L_f^k h](x, d, u)\} \neq 0 \right\} \quad i = 1, 2, \ldots, p \quad (5)$$

The relative degrees characterize how directly the inputs affect the output. The lower the relative degree, the more direct the effect of the input on the output.

To compare the relative effects of the manipulated input and the disturbances on the output, the disturbances can be categorized as follows (Daoutidis and Kravaris, 1989):

$$\begin{aligned} &1.\ d_{\mathcal{A}} \triangleq \{d_i : \rho_i > r\} \\ &2.\ d_{\mathcal{B}} \triangleq \{d_i : \rho_i = r\} \\ &3.\ d_{\mathcal{C}} \triangleq \{d_i : \rho_i < r\} \end{aligned} \quad (6)$$

Since they affect the output less directly than the manipulated input, the $d_{\mathcal{A}}$ disturbances can be decoupled from the output with feedback control. The $d_{\mathcal{B}}$ disturbances affect the output as directly as the manipulated input and can be decoupled from the output using feedback/feedforward control. The $d_{\mathcal{C}}$ disturbances affect the output more directly than the input and, therefore, cannot be decoupled from the output unless the control law contains derivatives of the disturbances (Daoutidis and Kravaris, 1989). Since measured disturbances are invariably corrupted with high-frequency noise, this approach is not considered here. In the subsequent development, we assume that $d_{\mathcal{B}}$ disturbances are measured and $d_{\mathcal{C}} = \{0\}$. If these conditions are not satisfied, nominal values for the $d_{\mathcal{B}}$ and $d_{\mathcal{C}}$ disturbances are used.

Design Method 1

The first design approach is based directly on the model in Eq. 2. If the two assumptions about the disturbances are satisfied, the first r time derivatives of y can be expressed as:

$$y^{(k)} = [L_f^k h](x) \quad k = 1, 2, \ldots, r - 1 \quad (7)$$

$$y^{(r)} = [L_f^r h](x, d_{\mathcal{B}}, u) \quad (8)$$

The first $r - 1$ derivatives are independent of u by the definition of the relative degree. If the nonlinear algebraic equation,

$$[L_f^r h](x, d_{\mathcal{B}}, u) = v \quad (9)$$

can be solved for u, then it follows from Eq. 8 that the map from the new input v to the output is linearized:

$$y^{(r)} = v \quad (10)$$

According to the implicit function theorem (Boothby, 1986), there exists a static feedforward-feedback control law,

$$u = \gamma(x, d_{\mathcal{B}}, v) \quad (11)$$

that is the solution of Eq. 9 if and only if:

$$(1)\ \frac{\partial}{\partial u} \{[L_f^r h](x, d_{\mathcal{B}}, u)\} \neq 0 \quad (12)$$

(2) For every x_o, $d_{\mathcal{B},o}$, and v_o there exists a u_o such that

$$[L_f^r h](x_0, d_{\mathcal{B},o}, u_o) = v_o \quad (13)$$

The first condition is always satisfied by definition of the relative degree of the manipulated input in Eq. 4. The second condition may or may not be satisfied. In general, the control law in Eq. 11 must be obtained numerically from Eq. 9.

The new input v is chosen as follows,

$$v = -\alpha_r y^{(r-1)} - \alpha_{r-1} y^{(r-2)} - \cdots - \alpha_1 y + \alpha_0 \int_0^t (y_{sp} - y)\, d\tau \quad (14)$$

where y_{sp} is the desired setpoint. Since the first $r - 1$ derivatives of y are functions of x only, Eq. 14 can be rewritten as:

$$v = -\alpha_r [L_f^{r-1} h](x) - \alpha_{r-1} [L_f^{r-2} h](x) - \cdots - \alpha_1 y + \alpha_0 \int_0^t (y_{sp} - y)\, d\tau \quad (15)$$

Thus the control law does not require derivatives of the output. The resulting closed-loop transfer function (CLTF) for setpoint changes is obtained from Eqs. 10 and 14:

$$\frac{y(s)}{y_{sp}(s)} = \frac{\alpha_0}{s^{r+1} + \alpha_r s^r + \cdots + \alpha_1 s + \alpha_0} \quad (16)$$

The controller in Eqs. 11 and 15 involves $r + 1$ tuning parameters. As suggested by Kravaris and Wright (1989), a CLTF with a single tuning parameter ϵ,

$$\frac{y(s)}{y_{sp}(s)} = \frac{1}{(\epsilon s + 1)^{r+1}} \quad (17)$$

can be obtained by choosing,

$$\alpha_k = \frac{(r + 1)\, r\, (r - 1) \cdots (r - k + 2)}{k!} \epsilon^{k-r-1} \quad 1 \leq k \leq r \quad (18)$$

where $\alpha_0 \triangleq \epsilon^{-(r+1)}$ by convention. Small values of ϵ will result in vigorous control action while large values will cause more sluggish responses. The integral action in Eq. 15 provides offset-free performance if modeling errors and/or unmeasured disturbances are present.

Design Method 2

The second design approach for Eq. 2 is based on the development of an "extended" model which is linear in a new manipulated input w. The input w is defined to be the time derivative of u (van der Schaft, 1984):

$$w \triangleq \dot{u} \quad (19)$$

The system in Eqs. 2 and 19 can then be written as an extended system which is in control linear form,

$$\dot{\bar{x}} = \bar{f}(\bar{x}, d) + gw$$
$$y = h(\bar{x}) \tag{20}$$

where

$$\bar{x} = \begin{bmatrix} x \\ u \end{bmatrix}, \bar{f}(\bar{x}, d) = \begin{bmatrix} f(x, d, u) \\ 0 \end{bmatrix}, g = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{21}$$

Due to the addition of the integrator to the input channel in Eq. 19, it is obvious that the relative degree of the new manipulated input w is equal to $r + 1$. Note that the relative degree ρ_i of the ith disturbance is not affected by the integrator. Three classes of disturbances, $\bar{d}_{\mathcal{A}}$, $\bar{d}_{\mathcal{B}}$, and $\bar{d}_{\mathcal{C}}$, can be defined for the extended system as in Eq. 6. Note that $\bar{d}_{\mathcal{C}} = \{d_{\mathcal{B}}, d_{\mathcal{C}}\}$ and $\{\bar{d}_{\mathcal{A}}, \bar{d}_{\mathcal{B}}\} = d_{\mathcal{A}}$. Hence, a smaller class of disturbances can be decoupled from the output by feedback or feedforward-feedback control using this approach. We assume that $\bar{d}_{\mathcal{B}}$ disturbances are measured and $\bar{d}_{\mathcal{C}} = \{0\}$. Otherwise, nominal values for the $\bar{d}_{\mathcal{B}}$ and $\bar{d}_{\mathcal{C}}$ disturbances are used.

Since Eq. 20 is linear with respect to the manipulated input w, a static feedforward-feedback control law can be designed for the extended system (Kravaris and Chung, 1987; Daoutidis and Kravaris, 1989). However, in terms of the original process variables, the control law is dynamic:

$$\dot{u} = \frac{v - [L_{\bar{f}}^{r+1} h]\,(x, \bar{d}_{\mathcal{B}}, u)}{[L_g L_{\bar{f}}^{r} h]\,(x, u)} \tag{22}$$

The input v can be chosen similar to Eq. 15 to yield a CLTF of the form:

$$\frac{y(s)}{y_{sp}(s)} = \frac{1}{(\epsilon s + 1)^{r+2}} \tag{23}$$

Simulations Study

A schematic of the continuous stirred tank reactor (CSTR) system is shown in Figure 1. The system consists of two constant volume reactors cooled by a single coolant stream flowing in a cocurrent fashion. An irreversible, exothermic reaction, $A \rightarrow B$, occurs in the two tanks. The objective is to control the effluent concentration from the second tank, C_{A2}, by manipulating the coolant flow rate, q_c. The process model consists of four nonlinear ordinary differential equations:

$$\dot{C}_{A1} = \frac{q}{V_1}(C_{Af} - C_{A1}) - k_0 C_{A1} \exp\left(-\frac{E}{RT_1}\right) \tag{24}$$

$$\dot{T}_1 = \frac{q}{V_1}(T_f - T_1) + \frac{(-\Delta H) k_0 C_{A1}}{\rho C_p} \exp\left(-\frac{E}{RT_1}\right) + \frac{\rho_c C_{pc}}{\rho C_p V_1} q_c \left[1 - \exp\left(-\frac{hA_1}{q_c \rho_c C_{pc}}\right)\right](T_{cf} - T_1) \tag{25}$$

$$\dot{C}_{A2} = \frac{q}{V_2}(C_{A1} - C_{A2}) - k_0 C_{A2} \exp\left(-\frac{E}{RT_2}\right) \tag{26}$$

$$\dot{T}_2 = \frac{q}{V_2}(T_1 - T_2) + \frac{(-\Delta H) k_0 C_{A2}}{\rho C_p} \exp\left(-\frac{E}{RT_2}\right) + \frac{\rho_c C_{pc}}{\rho C_p V_2} q_c \left[1 - \exp\left(-\frac{hA_2}{q_c \rho_c C_{pc}}\right)\right] \cdot \left[T_1 - T_2 + \exp\left(-\frac{hA_1}{q_c \rho_c C_{pc}}\right)(T_{cf} - T_1)\right] \tag{27}$$

If the state variables, disturbances, input, and output are defined as,

$$x \triangleq [C_{A1}\, T_1\, C_{A2}\, T_2]^T, \quad d \triangleq [C_{Af}\, T_{cf}]^T, \quad u \triangleq q_c, \quad y \triangleq C_{A2} \tag{28}$$

the model can be represented as in Eq. 2.

The model is not control linear, because the input variable q_c appears nonlinearly in Eqs. 25 and 27. In previous applications of differential geometric techniques to chemical reactors, the coolant temperature was chosen as the manipulated variable and assumed to be constant throughout the cooling coil (Hoo and Kantor, 1985; Kravaris and Chung, 1987; Calvert and Arkun, 1988). In our model, the coolant flow rate is chosen to be the manipulated variable and the coolant temperature is allowed to vary along the length of the cooling coil (Aris, 1969).

Open-loop responses for $\pm 10\%$ changes in the coolant flow rate are shown in Figure 2. The nominal operating conditions for the reactor are given in Table 1. Since the process gain is

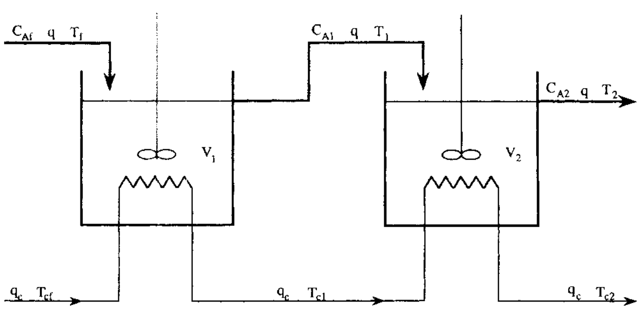

Figure 1. Continuous stirred tank reactor system.

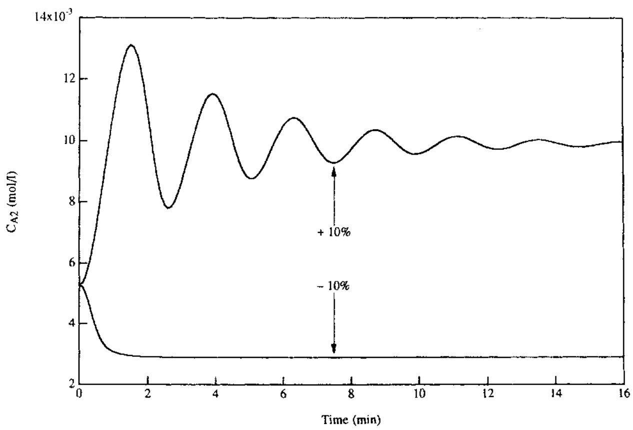

Figure 2. Open-loop composition response for ±10% changes in the coolant flow rate.

Table 1. Nominal Parameter Values for CSTR Model

$q = 100$ L/min	$k_o = 7.2 \times 10^{10}$ min^{-1}
$C_{Af} = 1$ mol/L	$E/R = 1 \times 10^4$ K
$T_f = 350$ K	$(-\Delta H) = 4.78 \times 10^4$ J/mol
$T_{cf} = 350$ K	$\rho = \rho_c = 1{,}000$ g/L
$V_1 = V_2 = 100$ L	$C_p = C_{pc} = 0.239$ J/g·K
$hA_1 = hA_2 = 1.67 \times 10^5$ J/min·K	

4.62×10^{-4} mol · min/L^2 for the +10% change and 2.41×10^{-4} mol · min/L^2 for the −10% change, the model exhibits significant static nonlinearities in this operating region. The dynamic behavior also illustrates the effect of the nonlinearity, as exemplified by the oscillation for the +10% change.

For comparison, three controllers were designed: a conventional proportional-integral (PI) controller, a static nonlinear controller (method 1), and a dynamic nonlinear controller (method 2). For simplicity, it was assumed that all four state variables can be measured. The PI controller was tuned to provide a compromise for two setpoint changes using settings of $K_c = 350$ L^2/mol · min and $\tau_I = 0.25$ min (Henson and Seborg, 1989). The static nonlinear controller was designed for the original nonlinear model in Eqs. 24–27 and implemented by solving a nonlinear algebraic equation of the form of Eq. 9. The dynamic nonlinear controller was designed for an extended model as in Eq. 20 and has the form of Eq. 22. The closed-loop transfer functions for the static and dynamic nonlinear controllers are of the form in Eqs. 17 and 23, respectively, with $r = 2$. Both nonlinear controllers were designed with $\epsilon = 0.25$ min. For

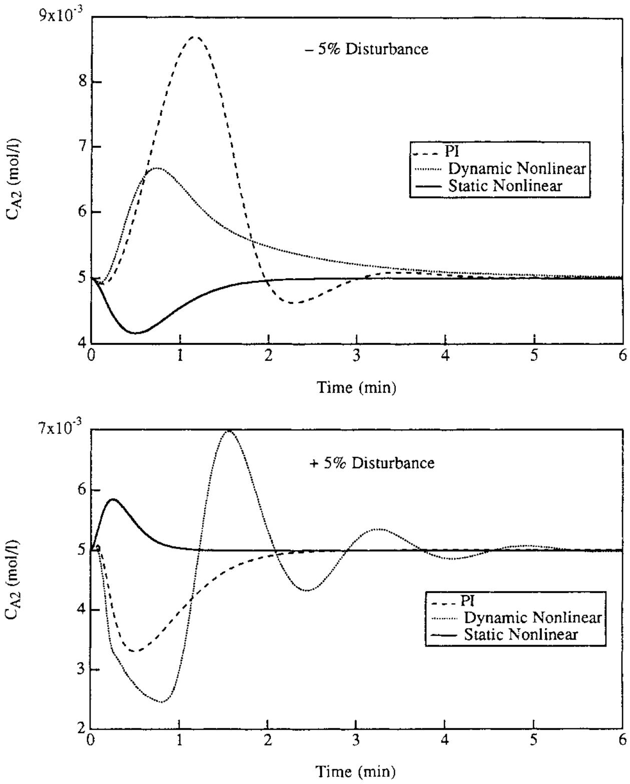

Figure 3. PI, static nonlinear, and dynamic nonlinear control for ±5% disturbances in the feed composition.

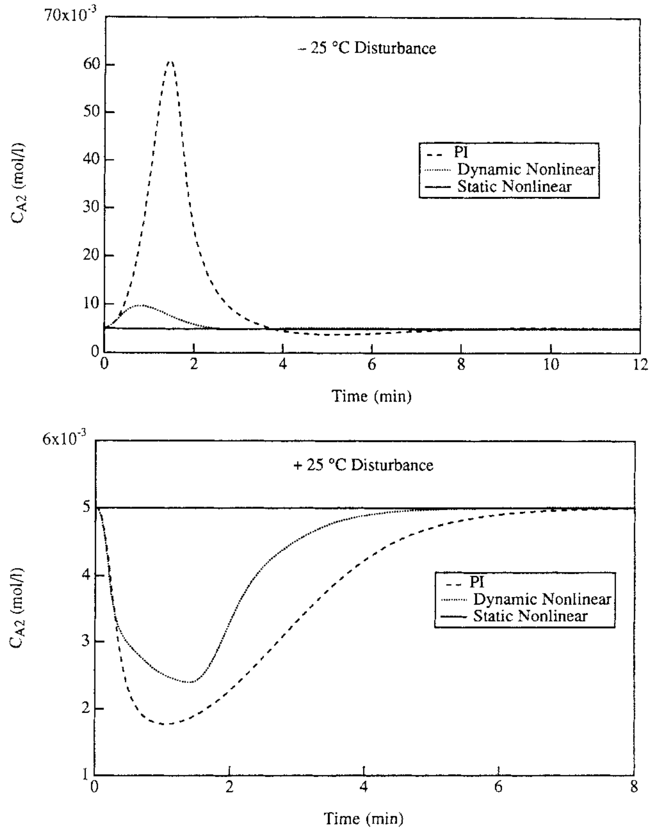

Figure 4. PI, static nonlinear, and dynamic nonlinear control for ±25°C disturbances in the feed temperature.

the two setpoint changes, the nonlinear controllers provided superior tracking without requiring large control actions (Henson and Seborg, 1989).

The disturbance rejection capabilities of the three controllers for ±5% unmeasured disturbances in the feed composition C_{Af} are shown in Figure 3. The C_{Af} disturbances cannot be decoupled from the output by either nonlinear controller because $\rho = 1$.

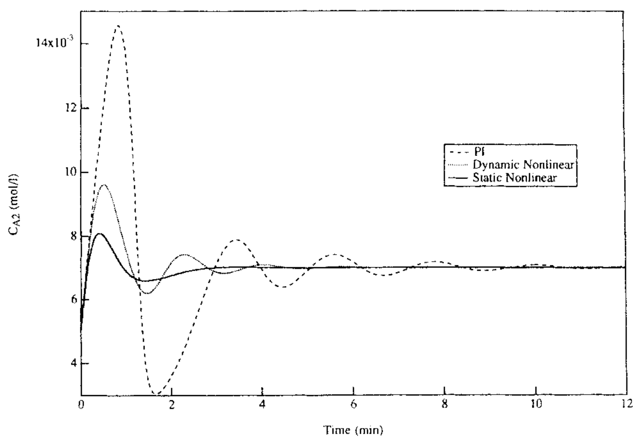

Figure 5. PI, static nonlinear, and dynamic nonlinear setpoint tracking for −20% error in the reaction rate.

However, the static nonlinear controller provides good control and is superior to the other two controllers for both C_{Af} disturbances. Somewhat surprisingly, the PI controller rejects the +5% disturbance much better than does the dynamic nonlinear controller.

The regulatory behavior of the three controllers for feed temperature T_f disturbances of ±25°C is shown in Figure 4. Unlike the C_{Af} disturbance, $\rho = 2$ for T_f. Because $r = 2$, a static feedforward-feedback control law of the form in Eq. 11 can be used to completely decouple the disturbance from the output. However, the results in Figure 4 for feedback control indicate that feedforward action is not required for the static controller. The static controller performs much better than the dynamic controller. Both nonlinear controllers are superior to the PI controller, especially for the −25°C disturbance.

The differences between the performance of the static and dynamic nonlinear controllers can be justified in terms of relative degrees. Results for the two unmeasured disturbances and several others not included in this note indicate that the difference between the relative degree of the manipulated input and the relative degree of the disturbance is important. It appears that the smaller this difference, the easier it is to reject the unmeasured disturbance. Note that this difference is always smaller for the static nonlinear controller than for the dynamic controller. Thus, the simulation results indicate that a static nonlinear controller based on the solution of a nonlinear algebraic equation exhibits superior disturbance rejection properties compared to a dynamic nonlinear controller based on an extended model. Moreover, a larger class of disturbances can be decoupled using a static controller. If the difference between the relative degrees of the manipulated input and disturbance is sufficiently small, both nonlinear controllers generally appear to provide superior performance to a conventional PI controller.

The robustness of the three controllers to an unanticipated change in the process is demonstrated in Figure 5, which shows the setpoint tracking behavior of the controllers for a −20% error in the reaction rate (+2% error in the activation energy). Although the dynamic controller had to be retuned (with $\epsilon = 0.1$ min) to handle the modeling error, the static controller still provided superior performance. The PI controller was stable, but performed poorly. Other simulation studies for a −25% error in the heat of reaction indicated that both nonlinear controllers outperformed the PI controller. The static nonlinear controller easily provided the best control.

Thus it appears that the static nonlinear controller is also superior to the dynamic nonlinear and PI controllers in terms of robustness. To explain the effects of parametric modeling errors on the performance of the static and dynamic nonlinear controllers, similar arguments to those developed for unmeasured disturbances can be developed. In terms of relative degrees, the more directly the modeling error affects the output, the more likely it is that the closed-loop performance will be degraded.

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Notation

A_i = heat transfer area for ith tank
C_A = concentration of component A
C_p = heat capacity
d = disturbance vector
E = activation energy
f = vector field for process model
h = heat transfer coefficient
h = nonlinear output equation
k_o = preexponential factor
$L_f^k h$ = kth-order Lie derivative of $h(x)$ with respect to $f(x)$
q = flow rate
r = relative degree of the input
T = temperature
u = manipulated input
V_i = volume of ith tank
v = manipulated input of input–output linearized model
w = manipulated input of extended model
x = state vector
$\bar{x}$ = state vector of extended model
y = output
y_{sp} = setpoint

Greek letters

ΔH = heat of reaction
ρ = density of reactor contents
ρ_i = relative degree of the ith disturbance

Subscripts

c = coolant

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